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Exact solutions of a nonlinear dispersive–dissipative equation

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Abstract. Within the invariant formalism of the Painlevé analysis, we have derived new exact solutions of a nonlinear dispersive–dissipative equation, which describes weak nonlinear ion-acoustic waves in a plasma consisting of cold ions and warm electrons, in terms of Bessel functions. Furthermore, a travelling wave solution is also obtained in a particular case.

1. Introduction

In recent years, the truncation procedure in Painlevé analysis has been widely used to derive special solutions of numerous nonlinear evolution equations. However, these solutions were obtained either by assuming the invariants C and S (see their definition below) are constant (Cariello and Tabor 1989, Guo and Chen 1991, Pickering 1993) or by proving their constance (Conte and Musette 1989).

In this paper, by solving the set of Painlevé–Bäcklund equations, we obtain an explicit form of both C and S and profit from the linearization property of the Riccati equations satisfied by the expansion variable to derive exact solutions. The equation we consider herein is

$$E \equiv u_t + uu_x + bu_{xxx} - a(u_t + muu_x)_x = 0 \tag{1}$$

where subscripts denote partial derivatives. Equation (1) has been derived by Kakutani and Kawahara (1970) by analysing a two-fluid plasma model, consisting of cold ions and warm electrons. As far as we know, no exact solutions of (1) have been derived until now. Recently, Malfliet (1994) has derived a travelling wave solution of (1) for $m = 0$ with the tanh method (Malfliet 1992). It is worth mentioning that the case $m \neq 0$ may also be treated with the same method to retrieve the kink-shaped travelling wave (23).

2. Painlevé analysis

We now apply the invariant Painlevé analysis to equation (1) and derive its special solutions. First we recall the main ideas of the invariant formalism of Painlevé analysis (Conte 1989).

Given a movable singular manifold

$$\phi - \phi_0 = 0 \tag{2}$$

the expansion variable χ, which must vanish as $\phi - \phi_0$, is chosen to be

$$\chi = \frac{\psi}{\psi_x} = \left(\frac{\phi_x}{\phi - \phi_0} - \frac{\phi_{xx}}{2\phi_x}\right)^{-1} \qquad \psi = (\phi - \phi_0)\phi_x^{\frac{1}{2}}. \tag{3}$$

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The variable χ satisfies the Riccati equations

$$\chi_x = 1 + \tfrac{1}{2} S\chi^2 \tag{4a}$$

$$\chi_t = -C + C_x\chi - \tfrac{1}{2}(CS + C_{xx})\chi^2 \tag{4b}$$

and the variable ψ satisfies the linear equations

$$\psi_{xx} = -\tfrac{1}{2} S\psi \tag{5a}$$

$$\psi_t = \tfrac{1}{2} C_x\psi - C\psi_x \tag{5b}$$

the coefficients of which are linked by the cross-derivative condition

$$S_t + C_{xxx} + 2C_x S + CS_x = 0. \tag{6}$$

S (Schwarzian derivative of ϕ) and C are defined by

$$S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \qquad C = -\frac{\phi_t}{\phi_x} \tag{7}$$

and are invariant under the group of homographic transformations

$$\phi \to \frac{a\phi + b}{c\phi + d} \qquad ad - bc = 1. \tag{8}$$

In this formalism, one looks for the general solution of (1) in the form

$$u = \chi^{-\alpha} \sum_{j=0}^{\infty} u_j \chi^j. \tag{9}$$

The functions u_j have to be determined by substitution of (9) into (1) which becomes

$$\sum_{j=0}^{\infty} E_j \chi^{j-\beta} = 0. \tag{10}$$

The leading-order analysis gives $\alpha = 1$, $u_0 = -2b/d$, $\beta = 4$ and $\{-1, 2, 3\}$ as indices, where $d = am$. It can easily be checked that (1) does not pass the Painlevé test and is therefore presumably not integrable. However, one can still determine special solutions by the truncation procedure, i.e. we look for solutions of (1) in the form

$$u_T = u_0\chi^{-1} + u_1. \tag{11}$$

Substituting (11) in (1), the set of Painlevé–Bäcklund equations to be solved for the validity of truncation (11) are

$$j = 1: \qquad b - d^2u_1 + adC = 0 \tag{12a}$$

$$j = 2: \qquad 2aC_x - C + u_1 - 2du_{1,x} = 0 \tag{12b}$$

$$j = 3: \qquad bS - d^2Su_1 + adCS + adC_{xx} + du_{1,x} - dC_x - d^2u_{1,xx} = 0 \tag{12c}$$

$$\begin{aligned} j = 4: \qquad & -dE(u_1) + bCS + bC_{xx} - b^2S_{xx} + bdS_xu_1 \\ & -2abSC_x - abCS_x - abC_{xxx} + 2bdSu_{1,x} - bSu_1 = 0. \end{aligned} \tag{12d}$$

Solving the set of equations (12*a*–*d*) together with the compatibility condition (6), we get

$$C = \frac{b}{d^2 - ad} \tag{13a}$$

$$u_1 = C \tag{13b}$$

and

$$dS_{xx} - S_x = 0 \tag{14a}$$

and

$$S_t + CS_x = 0. \tag{14b}$$

The general solution of (14*a*,*b*) is

$$S = \mu + \eta \exp\left(\frac{1}{d}(x - Ct)\right) \tag{15}$$

where μ and η are arbitrary constants.

In view of an explicit form of the solution (11), we have to determine χ which now satisfies, when C is a constant defined by (13*a*),

$$\chi_t + C\chi_x = 0. \tag{16}$$

We get over this problem by solving the linearized equation (5*a*), with S given by (15), which is now an ordinary differential equation of Bessel type (Abramowitz and Stegun 1964) where t acts as a parameter, i.e.

$$\psi_{xx} + \frac{1}{2}\left[\mu + \eta \exp\left(\frac{1}{d}(x - Ct)\right)\right]\psi = 0. \tag{17}$$

We distinguish three cases:

(i) $\mu\eta \neq 0$. Equation (17) has the following general solution

$$\psi = AJ_\nu(\lambda e^z) + BJ_{-\nu}(\lambda e^z) \qquad z = \frac{1}{2d}(x - Ct) \tag{18}$$

where J_ν and $J_{-\nu}$ are Bessel functions of order ν, $\lambda = 2d\sqrt{\eta/2}$, $\nu = 2d\sqrt{-\mu/2}$, A and B are arbitrary constants.

Therefore $\chi^{-1} = \psi_x/\psi$ depends on three arbitrary parameters λ, ν and the ratio A/B, i.e.

$$\chi^{-1} = \frac{\psi_x}{\psi} = \frac{\lambda}{2d}e^z \frac{[(A/B)J'_\nu(\lambda e^z) + J'_{-\nu}(\lambda e^z)]}{(A/B)J_\nu(\lambda e^z) + J_{-\nu}(\lambda e^z)} \tag{19}$$

where the prime stands for the derivative with respect to the argument. The final solution is then obtained from (11).

(ii) $\mu = 0$. The solution ψ is now a combination of Bessel and Neumann functions (J and N) of order zero:

$$\psi = AJ_0(\lambda e^z) + BN_0(\lambda e^z). \tag{20}$$

A similar solution to (19) is then obtained with ν replaced by zero.

(iii) $\eta = 0$. The solution is a kink generated by

$$\psi = Ae^{\gamma(x-Ct)} + Be^{-\gamma(x-Ct)} \qquad \gamma = \sqrt{\frac{-\mu}{2}}. \tag{21}$$

Therefore,

$$\chi^{-1} = \frac{\psi_x}{\psi} = \gamma \tanh(\gamma(x - Ct + x_0)) \qquad x_0 = \frac{1}{2\gamma}\ln(A/B). \tag{22}$$

A kink-shaped travelling wave is then obtained from (11), (13) and (22) in the form

$$u(x,t) = -\frac{2b}{d}\gamma \tanh\left[\gamma\left(x - \frac{b}{d(d-a)}t + x_0\right)\right] + \frac{b}{d(d-a)}. \tag{23}$$

3. Conclusion

Besides the usual travelling wave solutions obtained when the invariants S and C are constant, we have shown that another type of interesting solutions may be obtained for non-constant values of S. Other systems possess solutions to the truncation with S and C non-constant; one such example is the KPP equation (Conte 1988). Furthermore, Painlevé transcendents have particular solutions expressible in terms of classical transcendental functions; in particular P_{III} is related to the Bessel functions (Lukashevich 1965, 1967). Particular solutions in terms of Bessel functions have also been used to describe the so-called modons (Drazin and Johnson 1989).

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